Short- and Long-Term Dynamics of a Bead Constrained to a Rotating Circular Track

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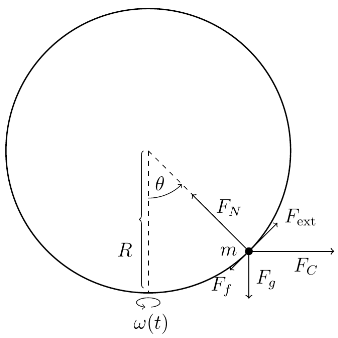
1 Abstract



We explore the problem of a bead constrained to a circular path under the effect of rotation about a fixed vertical axis. To analyze the problem, we used techniques associated with fictitious centrifugal forces from the rotating reference frame, non-conservative forces, and stable and unstable equilibria. We first consider the problem theoretically, both analytically and computationally, and then compare predictions from this model to experimental results. For the experimental model, we made use of the Invention Studio and constructed a disk out of acrylic with a channel carved out, whose width and depth were just enough to admit the plastic bead with minimal contact. We then hung the disk from a string, taped a neodymium magnet to it, and caused it to spin with a magnetic stir plate. In the course of our analysis, it became clear that the role of friction as a dissipative force is crucial to the accurate modeling and analysis of the system and leads to the observed phenomenon that all bead trajectories (other than those which begin and stay precisely in an unstable equilibrium) will always eventually converge to a stable equilibrium, regardless of the initial kinetic or gravitational potential energy given to the bead.

2 Introduction

This problem with constant angular rotation ω is well-explored in standard undergraduate nonlinear dynamics texts such as Strogatz [2] as it is relatively fundamental in the subject, and there is relatively little experimental research, although some related problems [1] have been studied in the literature.

The problem has a relatively simple premise, but it gives rise to a very highly nonlinear equation of motion which we analyze computationally. The external (non-friction) forces acting on the bead are gravity, the centrifugal force, and the normal force, expressed in cylindrical coordinates as

$$\boldsymbol{F}_{\text{ext}} = -mg\hat{\boldsymbol{z}} + m\omega^2 R\sin(\theta)\hat{\boldsymbol{r}} + \boldsymbol{F}_N$$

However, the problem is much more directly treated in spherical coordinates (flipped upside-down with the transformation $\theta \mapsto \pi - \theta$, so that small angles refer to the bottom of the ring), as the polar angle θ describes the bead's motion in the plane of the ring and the normal force is necessarily along the radial $\hat{\boldsymbol{\rho}}$ direction, so by a change of basis we see that

$$\boldsymbol{F}_{\text{ext}} = \left[mg\sin(\theta) - m\omega^2 R\sin(\theta)\cos(\theta)\right]\hat{\boldsymbol{\theta}} + \left[mg\cos(\theta) + m\omega^2 R\sin^2(\theta) + F_N\right]\hat{\boldsymbol{\rho}}$$

Since the bead is bound to a certain sphere at all times, the normal force must be such that the net radial acceleration is always zero. Then the net angular acceleration of the bead is

$$\ddot{\boldsymbol{\theta}}_{\text{net}} = \frac{1}{mR}\left(\boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_f\right) = \left[\left(\frac{g}{R} - \omega^2\cos(\theta)\right)\sin(\theta) - (\text{friction term})\right]\hat{\boldsymbol{\theta}}$$

Although we mentioned the constant-ω case, this equation applies equally when the ring rotates at some variable speed $\omega(t)$. We also don't write the friction term as any of μF_N, $bR\dot{\theta}$, $bR^2\dot{\theta}^2$ yet because it's not clear *a priori* which behavior will dominate. However, even with constant ω or without the friction term, this nonlinear ordinary differential equation can only be solved computationally, which we did in MATLAB using a simple Euler's method model. The predictions from this model will be discussed at length in conjunction with and in comparison to the experimental results. One thing that we will note beforehand, however, is that if the bead is stationary and friction plays no role, then the net force vanishes at the bottom and top of the ring and also at a pair of points $\theta = \pm\arccos\left(\frac{g/R}{\omega^2}\right)$, which are physical values with magnitude between 0 and $\frac{\pi}{2}$ for sufficiently large ring angular velocity ω.

3 Methods

Our construction consisted of materials that, admittedly, were chosen for ease of use in measuring data and machining with our limited tools. The original problem described a situation involving a bead threaded on a loop, but we realized that this setup would produce some potential limits. Firstly, the loop would have to be attached at some point to another apparatus in order to spin it, and the bead would not be able to traverse through this point, and secondly, the logistics of finding a low-friction bead and hoop system proved to be challenging. However, as long as some bead was still constrained to circular motion, as it is in the track we made in the end, the equations of motion should be identical- or at least mostly similar.

We constructed the track's housing in such a way that a string could be connected to one end, allowing it to spin along the string's axis. The string was fixed at a position in the disk close to the track in order to minimize external effects of the motion. The bead in question was colored a high contrast color in order to aid in motion tracking. A neodymium magnet was then taped to a position on the disk such that a magnetic stir plate would be able to spin the whole apparatus along the string axis.

To record the position of the bead, we set up a stand to record the apparatus using a phone's built-in high-speed camera. The phone recorded the video at 120 frames per second. We provided lighting on a white backdrop in order to provide contrast to the dark bead in order to process the video in MATLAB. We then ran trials, varying the angular velocity and tracking the dynamics of the bead.

In MATLAB, we processed the image by using a color mask and morphological operation to isolate the bead. We then utilized a circle tracking software, set up a spatial axis, and extracted data on the angle of the bead with respect to our axis for further analysis.

Unfortunately, due to various restraints in our setup, we were not able to properly track the exact dynamics of the ball on a spinning track, though we were able to do so for a stationary ring. For the rotating rings, we manually estimated angular velocity by moving frame by frame through the video and then recorded the equilibrium height from there.

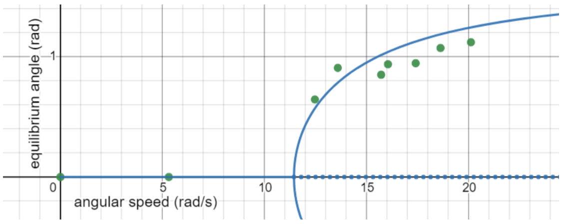
4 Results

We analyzed our data with the most thorough method each situation permitted. For the trials with non-zero rotational velocity, we estimated the exact rotational velocities for each situation, along with the equilibrium heights associated with those velocities.

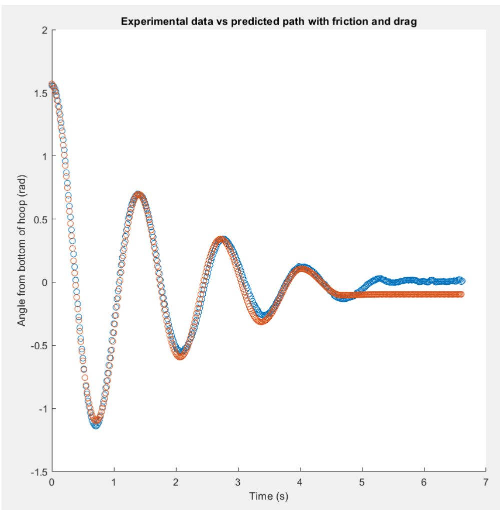
Radians Per Second	Equilibrium Height
2.654867031	0
6.231258156	1.5
6.792632765	2.843137255
7.853981634	2.586206897
8.021087626	3.103448276
8.699795041	3.214285714
9.308422677	4.111111111
10.05309649	4.111111111

As expected, the equilibrium heights start off at zero below a certain threshold, grow quickly at first, and taper off later. The general shape of the relationship is relatively consistent with the theory, including a sort of "dead zone" where the equilibrium height is 0 within a certain range of angular velocities.

Plotting our data with the theoretical curve yields the following



Additionally, we used the tracked motion of the stationary disk and some curve-fitting utility to estimate a value for the friction in the ball-track system, which yielded a coefficient of friction of about $\mu = 0.1$.



In the figure above, the red curve is the theoretical curve, and the blue one is the measured one.

5 Discussion

From our equations we estimated equilibrium height to be proportional to the inverse cosine of the negative square of the angular velocity:

$$\theta_{\text{eq}} = \arccos\left(\frac{\omega_0^2}{\omega^2}\right)$$

Our data, albeit a little noisy, seems to fit this general trend with the right coefficients added from our measured values (ring radius = 15cm, etc.). However, our values seem to trend a little flatter compared to the theoretical model. This could be due to a multitude of reasons, discussed a bit later.

In terms of the stationary-ring curve, the shape matches almost perfectly with our measured data until the end, where the theory separates a bit from the real world; in our theory, friction is assumed to be able to hold up the ball indefinitely at some angle not equal to 0 as seen in the curve, but this wouldn't be realistic with a physical model as the ball would simply roll down to the bottom of the track.

There are many potential sources of error from both a theory perspective, and an experimental perspective. A major issue we encountered throughout the project was due to the use of a string to hold up the apparatus. As the track spun, the tension in the winding would increase, gradually lifting the whole system up until the stable motion was lost; meaning we had a short window of time to gather data. Possible effects of this would be the gradual slowing of the angular velocity as the distance between the magnet and the stir plate increased, lowering the equilibrium angle, as well as the inability to test faster or more complex functions of angular velocity.

Additionally, camera position varied slightly throughout each trial simply due to human error. Since the virtual axes were defined consistently throughout each trial, this could have thrown off our measurements, contributing to the relatively noisy data.

We initially wanted to mount the camera to the rotating ring in order to properly track the evolution of the rotating beads, but due to the low torque provided from the magnetic stir plate and financial concerns (we used a phone to take the video), we decided against this notion and opted for a stationary camera instead. This, of course, only allowed us to track the stationary-disk system, and most likely led to a bit of inaccuracy, as the spinning-disk analysis was done mostly by hand.

Our theory also did not take into account that the ball was rolling across the track, rather, it was simply an object that experienced an opposing friction force as it moved along a track. As we can see in the static-disk curve, this leads to a few inconsistencies, but is comparatively, relatively minor.

The theory also did not account for the effects of air resistance. Since the track was relatively closed, the dampening effects on the bead were likely slightly significant. We attempted to alleviate this issue by adding small spacers between the tracks to allow air to escape while the bead rolled. Additionally, incorporating a dampening effect in our theory would most likely lead to a much harder problem. This dampening factor most likely inflated our estimated value of friction, if any effect at all.

6 Conclusion

In idealized situations, it is often simple to gain intuitive understanding about a system upon a first glance. However, as we have seen with our project, adding variables due to the real-world clearly not being a frictionless, air-resistance-less, ideal world, we have to account for many variables when considering dynamics of interacting systems which often lead to many unforseen consequences.

Our project successfully used theory to estimate a relationship between equilibrium heights of the bead-ring system, and built a simple physical apparatus to test our theories. The experimental data generally fit our theoretical curves, and confirmed many of the interesting dynamics that were predicted, such as the aforementioned "dead zone" in the equilibrium angle vs. angular speed data.

We also estimated a value for rolling friction between the bead and the acrylic used in the ring by using tracking software and curve-fitting utilities on a stationary-ring system.

We have not tested the full extent of this system and possible variables to include. Some possible interesting dynamics could arise from time-variant angular velocities, or tilting the whole system a certain amount, as in [1]. With a more refined setup, the introduction of additional variables such as the ones above could lead to much more unexpected behavior.

Though this problem seems relatively fundamental in terms of its setup, it could have many implications to systems constrained to circular orbit- planetary motion and orbital mechanics immediately come into mind. Additionally, it would be interesting to generalize our situation into the n^{th} dimensional case to analyze possible applications to higher-level physics.

References

[1] Lisandro A Raviola et al. "The bead on a rotating hoop revisited: An unexpected resonance". In: *European Journal of Physics* 38.1 (2016). DOI: 10.1088/0143-0807/38/1/015005.

[2] Steven Strogatz. "Overdamped Bead on a Rotating Hoop". In: *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. 2nd. CRC Press, Taylor et Francis Group, pp. 62–69.